Plateau–Gibbs Borders: Calculation of Shape and Volume

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The shape of the Plateau–Gibbs borders in a foam is described and the analytical equations for the correction coefficients for the volume of the borders are given.

A simple model for Plateau–Gibbs borders, in which the shape is approximated by the gap between three cylindrical surfaces with radius r_c, is usually used to study a foam's density at hydrostatic equilibrium,[1–3] in terms of electric conductivity[3–5] and to study foam syneresis.[3,6] According to this model, the cross-sectional area of a border in a cylindrical foam is $S_c = 0.16r_c^2$ and the volume $V_{cm} = S_c l_c$ (l_c = border length).

It is clear that the cylindrical model describes the border's properties only approximately: an experimental study of the shape of borders has been carried out[7] with the help of a model consisting of an inflatable rubber balloon inside a transparent dodecahedron. According to the main direction of our research, we have carried out a numerical solution of the Laplace equation in order to calculate the correction coefficients for the described model.

Spatial coordinates for the Plateau–Gibbs borders were assigned as follows. The abscissa x coincided with the axis of the border, the origin ($x = 0$) being located at the centre of the border, and the ordinate y was perpendicular to the surface of the border. Thus, the $x0y$ plane coincided with the plane of symmetry (Fig. 1). The shape of a 1/12 part of the border between the $x = 0$, $z = 0$ planes, the plane of the film (assigned at an angle of 60° relative to the $z = 0$ symmetry plane) and the plane perpendicular to that of the film was analysed. The values
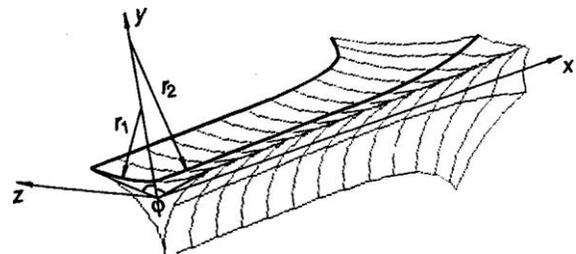

Fig. 1 Coordinate system for the Plateau–Gibbs border simulation (r_1 and r_2 are the diametrical and longitudinal radii of curvature, respectively); $\phi = 60°$

of the diametrical r_1^0 and longitudinal r_2^0 principle radii of curvature of the border surface were set as an initial condition. The average constant radius of curvature of the surface was calculated from the Laplace equation [eqn. (1)], where r_c is the

$$\bar{r} = 2r_c = 2(1/r_1 + 1/r_2)^{-1} = \text{Const.} \tag{1}$$

radius of curvature of the surface of a cylindrical border at the same capillary pressure. It is important to note that, as an

approximation, we considered extremely thin films, and at the same time wall effects and the influence of gravitational forces were ignored. According to these approximations, the structure of the foam (in particular the ratio r_1^0/r_2^0) is defined by the volume fraction of liquid in the foam (ϕ) and does not depend on the dispersity of the foam (in the same way as the degree of space-filling by close-packed spheres is independent of their size). The displacement Δx along the x-axis was also pre-set.

Analysis of the shape of the border was carried out on the basis of a calculation of the variation of the principal radii of curvature r_1, r_2 along the x-axis and the variation of the y-coordinates of the central line formed by the intersection of the border's surface with the $z = 0$ plane. The vertical displacement of the central line, Δy, and the change in its slope, $\Delta(dy/dx)$, were calculated for cycle i for displacement from x_{i-1} to x_i using the value of r_2 obtained for cycle $i - 1$. The determination of the new value of r_1 was based on the assumption that it could be identified with the radius of a sphere touching both the y (x) curve at point x and each of the two surfaces of the films. Thus, a new value of r_2 was calculated from the new r_1 using the Laplace equation [eqn. (1)].

The coordinates x_c, y_c, z_c of the point of contact of the sphere with the plane of the film and the coordinates of several intermediate points on a circle representing the cross-section cut through the sphere by a plane passing through its centre were calculated. The cross-sectional area S_i of the border under consideration and the projection of the displacement Δx to the perpendicular H_i of this plane were also calculated.

The portion of the volume of the border, ΔV_c, between the two successive sections was determined as the product of the average cross-sectional area the height (H_i) [eqn. (2)].

$$\Delta V_{ci} = H_i(S_{i-1} + S_i)/2 \tag{2}$$

It should be noted that the main source of errors in the calculation is, apparently, this method of determining the average distance H_i between neighbouring sections.

Calculations were carried out as long as the condition $\arctan(dy/dx) < 0.34$ was satisfied, the value 0.34 being the angle (in radians) of the side of a tetrahedron to its height.

The overall length of the border between the centres of the two bundles, l_c, was calculated from coordinates x_n, y_n of the last point [eqn. (3)].

$$l_c = 2(x_n + \tan 0.34\ y_n) \tag{3}$$

The volume of the border, V_c, and the ratio of V_c to the analogous volume in a cylindrical model (f_V) having the same value of l_c were determined as shown in eqns. (4) and (5). It is

$$V_c = 2\sum_{i=1}^{n} \Delta V_{ci} \tag{4}$$

$$f_V = V_c/V_{cm} \tag{5}$$

known[8] that the volume of a pentadodecahedron is $7.7l_c^3$. Each pentadodecahedron possesses 30 edges, each of which at the same time belongs to three cells. Hence, the volume of liquid in one cell is $10V_c$, where V_c is the volume of the border. The value of ϕ was estimated using the dodahedrical model [eqn. (6)].

$$\phi = 10V_c/7.7l_c^3 = 1.30V_c/l_c^3 \tag{6}$$

Calculations were carried out with $r_1^0 = 0.01$ and r_2^0 varying from 0.01 to 10^6. The results obtained can be described by eqn. (7), the relative error in the calculation being 7%.

$$f_V = 1 + 13.9\phi - 20.3\phi^2 \tag{7}$$

The shapes of the foam bundles shown in Figs. 2 and 3 resulted from a combination of the different shapes found for borders in foams with various densities. The computational error is pronounced in places where the surfaces of neighbouring borders come into contact. The results show that the surface of the border has a cylindrical shape along its entire length, but there is a sharp widening of the border near the edge. It is necessary to point out that the length of border is only a few
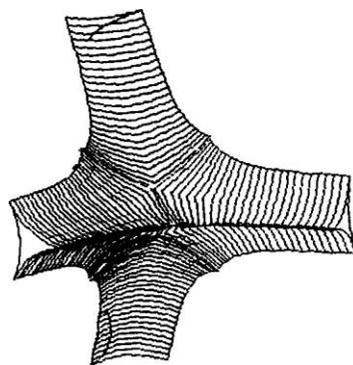

Fig. 2 Plateau–Gibbs border bundle of a foam with density $9.48 \times 10^{-3}\ \mathrm{g\ cm^{-3}}$
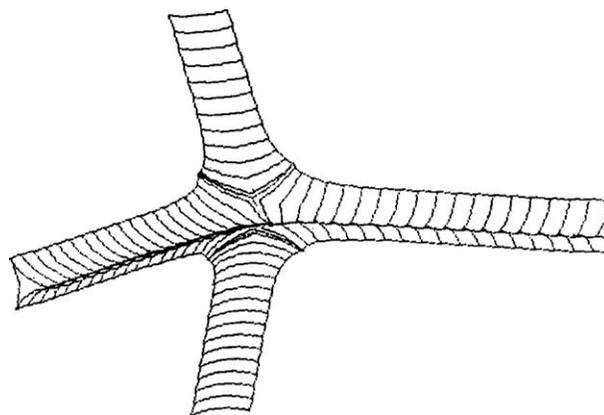

Fig. 3 Plateau–Gibbs border bundle of a foam with density $1.43 \times 10^{-3}\ \mathrm{g\ cm^{-3}}$

times larger than its diametrical radius r_1^0 and it is in many orders of magnitude smaller than the longitudinal radius r_2^0 at the centre of a border.

Analogous calculations were carried out to determine the correction coefficients for electrical conductivity and hydroconductivity.

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